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The expansion problem of anti-symmetric matrix under a linear constraint and the optimal approximation ☆

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Abstract

This paper mainly discusses the following two problems:

Problem I. Given $A \in R^{n\times m}$, $B \in R^{m\times m}$, $X_0 \in \mathrm{ASR}^{q\times q}$ (the set of $q \times q$ anti-symmetric matrices), find $X \in ASR^{n\times n}$ such that

$$A^{\mathrm{T}}XA = B, \quad X_0 = X([1:q]),$$

where $X([1:q])$ is the $q \times q$ leading principal submatrix of matrix X.

Problem II. Given $X^* \in R^{n\times n}$, find $\hat{X} \in S_E$ such that

$$\|X^* - \hat{X}\| = \min_{X\in S_E} \|X^* - X\|,$$

where $\|\cdot\|$ is the Frobenius norm, and S_E is the solution set of Problem I.

The necessary and sufficient conditions for the existence of and the expressions for the general solutions of Problem I are given. Moreover, the optimal approximation solution, an algorithm and a numerical example of Problem II are provided.

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Keywords: Anti-symmetric matrix; Linear constraint; Frobenius norm; Optimal approximation

1. Introduction

The notations in this paper are as follows: $R^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathrm{OR}^{n\times n}$ denotes the set of $n\times n$ real orthogonal matrices, $\mathrm{ASR}^{n\times n}$ denotes the set of $n\times n$ anti-symmetric matrices. $X([1:q])$, $\mathrm{rank}(X)$ and $\|X\|$

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denote the $q \times q$ leading principal submatrix, the rank and the Frobenius norm of matrix X, respectively. I_k denotes the $k \times k$ unit matrix. For matrices $A = (a_{ij})$, $B = (b_{ij}) \in R^{n\times m}$, $A * B \stackrel{\triangle}{=} (a_{ij}b_{ij}) \in R^{n\times m}$ denotes their Hadamard product.

The matrix expansion problem with a certain constraint in a certain set comes from a practical subsystem expansion problem. For some results on it, we refer the readers to [3,5,8,9,11,12]. For example, [3,5] studied the expansion problem of Jacobi matrix and tridiagonal matrix under a spectrum constraint. [9] considered the expansion problem of Jacobi matrix with a prescribed ordered defective eigenpairs constraint. [8,11,12] discussed the expansion problem of matrix X with a linear constraint $XA = B$ in the set of real matrices, centro-symmetric matrices and bisymmetric matrices respectively. The solvability conditions, expressions of general solutions and optimal approximation solution are provided in these papers.

$A^{\mathrm{T}}XA = B$ is another important linear constraint for matrix X. In fact, as a matrix equation of finding unknown X in a certain set, $A^{\mathrm{T}}XA = B$ comes from a practical vibration problem. As for this problem, some results can be found in [1,2,6,10]. However, to the expansion problem of anti-symmetric matrix X with a linear constraint $A^{\mathrm{T}}XA = B$, there is no relative discussion yet. And one purpose of this paper is to study this problem, that is

Problem I. Given $A \in R^{n\times m}$, $B \in R^{m\times m}$, $X_0 \in \mathrm{ASR}^{q\times q}$, find $X \in \mathrm{ASR}^{n\times n}$ such that

$$A^{\mathrm{T}}XA = B, \quad X_0 = X([1:q]),$$

where $X([1:q])$ is the $q \times q$ leading principal submatrix of matrix X.

The problem of finding the nearest matrix to a given matrix in a certain set is called the optimal approximation problem. It is initially proposed in the processes of test or recovery of linear systems due to incomplete dates or revising dates. There are many results about the optimal approximation problem, we refer the readers to [4,13–15] etc. And the other purpose of this paper is to solve the relative optimal approximation problem of Problem I, namely

Problem II. Given $X^* \in R^{n\times n}$, find $\hat{X} \in S_E$ such that

$$\|X^* - \hat{X}\| = \min_{X\in S_E} \|X^* - X\|,$$

where S_E is the solution set of Problem I.

The paper is organized as follows. First, we provide the solvability conditions of Problem I and its general solution in that case in Section 2. Then we prove that Problem II has an unique solution and get the expression of it in Section 3. In Section 4 we give an algorithm and a numerical example of Problem II. Finally, we make some conclusions in Section 5.

2. The solution of Problem I

As a preliminary, we state some results about SVD [16] and GSVD [7].

Given $A \in R^{n\times m}$, the singular value decomposition of matrix A is

$$A = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^{\mathrm{T}}, \tag{2.1}$$

where $U = (U_1, U_2) \in \mathrm{OR}^{n\times n}$, $V = (V_1, V_2) \in \mathrm{OR}^{m\times m}$, $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_i > 0 (i = 1, \ldots, r)$, $r = \mathrm{rank}(A)$, $U_1 \in R^{n\times r}$, $V_1 \in R^{m\times r}$.

Let

$$(I_q, 0)U = (\tilde{U_1}, \tilde{U_2}), \quad \tilde{U_1} \in R^{q\times r}, \quad \tilde{U_2} \in R^{q\times(n-r)}, \tag{2.2}$$

where U is defined by (2.1). Then the generalized singular-value decompositions of the matrix pair $[\tilde{U_1}^{\mathrm{T}}, \tilde{U_2}^{\mathrm{T}}]$ is

$$\tilde{U_1}^{\mathrm{T}} = P\Gamma_1 W^{\mathrm{T}}, \quad \tilde{U_2}^{\mathrm{T}} = Q\Gamma_2 W^{\mathrm{T}}, \tag{2.3}$$

where $P \in \mathrm{OR}^{r\times r}$, $Q \in \mathrm{OR}^{(n-r)\times(n-r)}$, $W \in R^{q\times q}$ is a nonsingular real matrix,

$$\Gamma_1 = \begin{array}{c} \begin{pmatrix} I_s & & & \vdots & \\ & \Omega_1 & & \vdots & O \\ & & O_1 & \vdots & \end{pmatrix} \\ \begin{array}{cccc} s & l & t-s-l & q-t \end{array} \end{array} \begin{array}{l} s \\ l \\ r-s-l \end{array}, \quad \Gamma_2 = \begin{array}{c} \begin{pmatrix} O_2 & & & \vdots & \\ & \Omega_2 & & \vdots & O \\ & & I_{t-s-l} & \vdots & \end{pmatrix} \\ \begin{array}{cccc} s & l & t-s-l & q-t \end{array} \end{array} \begin{array}{l} n-r-t+s \\ l \\ t-s-l \end{array}. \tag{2.4}$$

$t = \mathrm{rank}(\tilde{U}_1, \tilde{U}_2) = \mathrm{rank}((I_q, 0)U)$, $s = t - \mathrm{rank}(\tilde{U}_2)$, $l = \mathrm{rank}(\tilde{U}_1) + \mathrm{rank}(\tilde{U}_2) - t$. O, O_1, O_2 are zero matrices of suitable sizes, $\Omega_1 = \mathrm{diag}(a_1, a_2, \ldots, a_l) > 0$, $\Omega_2 = \mathrm{diag}(b_1, b_2, \ldots, b_l) > 0$.

The following lemma will be used in this section:

Lemma 2.1. *Given $A \in R^{n\times m}$, $B \in R^{m\times m}$, suppose the SVD of matrix A is given as* (2.1). *Then the matrix equation $A^{\mathrm{T}}XA = B$ has a solution $X \in \mathrm{ASR}^{n\times n}$ if and only if*

$$B^{\mathrm{T}} = -B, \quad BV_2 = 0.$$

And in that case, the general solution is

$$X = U \begin{pmatrix} \Sigma^{-1}V_1^{\mathrm{T}}BV_1\Sigma^{-1} & X_{12} \\ -X_{12}^{\mathrm{T}} & X_{22} \end{pmatrix} U^{\mathrm{T}},$$

where $X_{12} \in R^{r\times(n-r)}$, $X_{22} \in ASR^{(n-r)\times(n-r)}$ are arbitrary matrices.

Lemma 2.1 can be similarly obtained as Theorem 2.1 in [2].

Theorem 2.1. *Suppose matrices A, B and X_0 are given in Problem I, the SVD of matrix A is* (2.1), *matrix pair $[\tilde{U}_1{}^{\mathrm{T}}, \tilde{U}_2{}^{\mathrm{T}}]$ is defined in* (2.2), *and the GSVD of it has the form* (2.3) *and* (2.4). *Let*

$$\tilde{X} = W^{-1}(X_0 - \tilde{U}_1\Sigma^{-1}V_1^{\mathrm{T}}BV_1\Sigma^{-1}\tilde{U}_1^{\mathrm{T}})W^{-\mathrm{T}} = \begin{array}{c} \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} & \tilde{X}_{14} \\ \tilde{X}_{21} & \tilde{X}_{22} & \tilde{X}_{23} & \tilde{X}_{24} \\ \tilde{X}_{31} & \tilde{X}_{32} & \tilde{X}_{33} & \tilde{X}_{34} \\ \tilde{X}_{41} & \tilde{X}_{42} & \tilde{X}_{43} & \tilde{X}_{44} \end{pmatrix} \\ \begin{array}{cccc} s & l & t-s-l & q-t \end{array} \end{array} \begin{array}{l} s \\ l \\ t-s-l \\ q-t \end{array}, \tag{2.5}$$

then Problem I is solvable if and only if

$$\begin{cases} B^{\mathrm{T}} = -B, \quad BV_2 = 0, \\ \tilde{X}_{11} = 0, \quad \tilde{X}_{41} = 0, \quad \tilde{X}_{42} = 0, \quad \tilde{X}_{43} = 0, \quad \tilde{X}_{44} = 0. \end{cases} \tag{2.6}$$

And when (2.6) *holds, the general solution of Problem I is*

$$X = U \begin{pmatrix} \Sigma^{-1}V_1^{\mathrm{T}}BV_1\Sigma^{-1} & X_{12} \\ -X_{12}^{\mathrm{T}} & X_{22} \end{pmatrix} U^{\mathrm{T}}, \tag{2.7}$$

where

$$X_{12} = P \begin{pmatrix} Y_{11} & \tilde{X}_{12}\Omega_2^{-1} & \tilde{X}_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} Q^{\mathrm{T}},$$

$$X_{22} = Q \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ -Z_{12}^{\mathrm{T}} & \Omega_2^{-1}(\tilde{X}_{22} - \Omega_1Y_{22}\Omega_2 + \Omega_2Y_{22}^{\mathrm{T}}\Omega_1)\Omega_2^{-1} & \Omega_2^{-1}(\tilde{X}_{23} - \Omega_1Y_{23}) \\ -Z_{13}^{\mathrm{T}} & (Y_{23}^{\mathrm{T}}\Omega_1 - \tilde{X}_{23}^{\mathrm{T}})\Omega_2^{-1} & \tilde{X}_{33} \end{pmatrix} Q^{\mathrm{T}}.$$

where $Y_{11} \in R^{s\times(n-r-t+s)}$, $Y_{22} \in R^{l\times l}$, $Y_{33} \in R^{(r-s-l)\times(t-s-l)}$, $Z_{11} \in \mathrm{ASR}^{(n-r-t+s)\times(n-r-t+s)}$, $Z_{12} \in R^{(n-r-t+s)\times l}$, $Z_{13} \in R^{(n-r-t+s)\times(t-s-l)}$, Y_{21}, Y_{23}, Y_{31}, Y_{33}, are arbitrary matrices.

Proof. By Lemma 2.1 and (2.1), there exists $X \in \mathrm{ASR}^{n\times n}$ such that $A^{\mathrm T}XA=B$ if and only if

$$B^{\mathrm T}=-B, \quad BV_2=0. \tag{2.8}$$

And when (2.8) holds, the general expression of the anti-symmetric solution of matrix equation $A^{\mathrm T}XA=B$ is

$$X=U\begin{pmatrix}\Sigma^{-1}V_1^{\mathrm T}BV_1\Sigma^{-1} & X_{12}\\ -X_{12}^{\mathrm T} & X_{22}\end{pmatrix}U^{\mathrm T}, \tag{2.9}$$

where $X_{12}\in R^{r\times(n-r)}$, $X_{22}\in \mathrm{ASR}^{(n-r)\times(n-r)}$ are arbitrary matrices. By using $X_0=X([1:q])$ and substituting (2.2) into (2.9), we have

$$\tilde U_1X_{12}\tilde U_2^{\mathrm T}-\tilde U_2X_{12}^{\mathrm T}\tilde U_1^{\mathrm T}+\tilde U_2X_{22}\tilde U_2^{\mathrm T}=X_0-\tilde U_1\Sigma^{-1}V_1^{\mathrm T}BV_1\Sigma^{-1}\tilde U_1^{\mathrm T}. \tag{2.10}$$

Substituting (2.3) into (2.10) and notice (2.5), there is

$$\Gamma_1^{\mathrm T}P^{\mathrm T}X_{12}Q\Gamma_2-\Gamma_2^{\mathrm T}Q^{\mathrm T}X_{12}^{\mathrm T}P\Gamma_1+\Gamma_2^{\mathrm T}Q^{\mathrm T}X_{22}Q\Gamma_2=\tilde X. \tag{2.11}$$

Partitioning matrices $P^{\mathrm T}X_{12}Q$ and $Q^{\mathrm T}X_{22}Q$ into the following form:

$$P^{\mathrm T}X_{12}Q=\begin{array}{c@{}l}\begin{pmatrix}Y_{11} & Y_{12} & Y_{13}\\ Y_{21} & Y_{22} & Y_{23}\\ Y_{31} & Y_{32} & Y_{33}\end{pmatrix} & \begin{array}{l}s\\ l\\ r-s-l\end{array}\\ \begin{array}{ccc}n-r-t+s & l & t-s-l\end{array} & \end{array},$$

$$Q^{\mathrm T}X_{22}Q=\begin{array}{c@{}l}\begin{pmatrix}Z_{11} & Z_{12} & Z_{13}\\ -Z_{12}^{\mathrm T} & Z_{22} & Z_{23}\\ -Z_{13}^{\mathrm T} & -Z_{23}^{\mathrm T} & Z_{33}\end{pmatrix} & \begin{array}{l}n-r-t+s\\ l\\ t-s-l\end{array}\\ \begin{array}{ccc}n-r-t+s & l & t-s-l\end{array} & \end{array}. \tag{2.12}$$

and substituting (2.3), (2.4), (2.5), (2.12) into (2.11), we know

$$\begin{pmatrix}0 & Y_{12}\Omega_2 & Y_{13} & 0\\ -\Omega_2Y_{12}^{\mathrm T} & \Omega_1Y_{22}\Omega_2-\Omega_2Y_{22}^{\mathrm T}\Omega_1+\Omega_2Z_{22}\Omega_2 & \Omega_1Y_{23}+\Omega_2Z_{23} & 0\\ -Y_{13}^{\mathrm T} & -Y_{23}^{\mathrm T}\Omega_1-Z_{23}^{\mathrm T}\Omega_2 & Z_{33} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$
$$=\begin{pmatrix}\tilde X_{11} & \tilde X_{12} & \tilde X_{13} & \tilde X_{14}\\ \tilde X_{21} & \tilde X_{22} & \tilde X_{23} & \tilde X_{24}\\ \tilde X_{31} & \tilde X_{32} & \tilde X_{33} & \tilde X_{34}\\ \tilde X_{41} & \tilde X_{42} & \tilde X_{43} & \tilde X_{44}\end{pmatrix}. \tag{2.13}$$

Since X_0 is an anti-symmetric matrix, $\tilde X$ is obviously an anti-symmetric matrix too when (2.8) holds. Thus (2.13) holds if and only if

$$\tilde X_{11}=0,\quad \tilde X_{41}=0,\quad \tilde X_{42}=0,\quad \tilde X_{43}=0,\quad \tilde X_{44}=0, \tag{2.14}$$

$$\begin{cases}Y_{12}=\tilde X_{12}\Omega_2^{-1}, \quad Y_{13}=\tilde X_{13},\\ Z_{22}=\Omega_2^{-1}(\tilde X_{22}-\Omega_1Y_{22}\Omega_2+\Omega_2Y_{22}^{\mathrm T}\Omega_1)\Omega_2^{-1}, \quad Z_{23}=\Omega_2^{-1}(\tilde X_{23}-\Omega_1Y_{23}), \quad Z_{33}=\tilde X_{33}.\end{cases} \tag{2.15}$$

Combining (2.8) with (2.14), the necessary and sufficient conditions under which Problem I is solvable is (2.6). By substituting (2.15) into (2.12) then into (2.9), we know when (2.6) holds, the general solution of Problem I can be expressed as (2.7). □

3. The solution of Problem II

First we introduce some lemmas.

Lemma 3.1. *Given* $D_1, D_2, F_1, F_2 \in R^{m\times n}$, $\Lambda = \mathrm{diag}(a_1, a_2, \ldots, a_m) > 0$, *then*

(i) $\|G - D_1\|^2 + \|G - D_2\|^2 = \min$ *has an unique solution* $\hat{G} \in R^{m\times n}$ *which can be expressed as* $\hat{G} = \frac{1}{2}(D_1 + D_2)$.
(ii) $\|G - D_1\|^2 + \|G - D_2\|^2 + \|\Lambda G - F_1\|^2 + \|\Lambda G - F_2\|^2 = \min$ *has an unique solution* $\hat{G} \in R^{m\times n}$ *which can be expressed as*

$$\hat{G} = \Phi * (D_1 + D_2 + \Lambda F_1 + \Lambda F_2),$$

where $\Phi = (\varphi_{ij}) \in R^{m\times n}$, $\varphi_{ij} = 1/2(1 + a_i^2)$.

Lemma 3.2. *Given* $D_1, D_2, F \in R^{n\times n}$, $\Lambda = \mathrm{diag}(a_1, a_2, \ldots, a_n) > 0$, *then*

(i) $\|G - F\|^2 = \min$ *has an unique solution* $\hat{G} \in \mathrm{ASR}^{n\times n}$ *which can be expressed as* $\hat{G} = \frac{1}{2}(F - F^{\mathrm{T}})$.
(ii) $\|G - D_1\|^2 + \|G - D_2\|^2\| + \|\Lambda G - G^{\mathrm{T}}\Lambda - F\|^2 = \min$ *has an unique solution* $\hat{G} \in R^{n\times n}$ *which can be expressed as*

$$\hat{G} = \Phi * (D_1 + D_2 + D_1\Lambda^2 + D_2\Lambda^2 + \Lambda D_1^{\mathrm{T}}\Lambda + \Lambda D_2^{\mathrm{T}}\Lambda + \Lambda(F - F^{\mathrm{T}})),$$

where $\Phi = (\varphi_{ij}) \in R^{n\times n}$, $\varphi_{ij} = 1/2(1 + a_i^2 + a_j^2)$.

To prove Lemmas 3.1 and 3.2, we just need to develop the functions of G into the square polynomials about g_{ij}, then by using the character of G and the differential theory we can obtain the results. Here, we only provide the proof of Lemma 3.2(ii).

Proof. Suppose $G = (g_{ij})_{n\times n}$, $D_1 = (d_{1ij})_{n\times n}$, $D_2 = (d_{2ij})_{n\times n}$, $F = (f_{ij})_{n\times n}$, then

$$\begin{aligned} f(G) &= \|G - D_1\|^2 + \|G - D_2\|^2 + \|\Lambda G - G^{\mathrm{T}}\Lambda - F\|^2 \\ &= \sum_{1\leqslant i,j\leqslant n} \{(g_{ij} - d_{1ij})^2 + (g_{ij} - d_{2ij})^2 + (a_i g_{ij} - a_j g_{ji} - f_{ij})^2\}. \end{aligned}$$

Notice that the minimization of $f(G)$ is equivalent to

$$\begin{aligned} \frac{\partial f(G)}{\partial g_{ij}} &= 2(g_{ij} - d_{1ij}) + 2(g_{ij} - d_{2ij}) + 2a_i(a_i g_{ij} - a_j g_{ji} - f_{ij}) \\ &\quad - 2a_i(a_j g_{ji} - a_i g_{ij} - f_{ji}) = 0, \quad 1\leqslant i, j\leqslant n. \end{aligned}$$

From that we have

$$\begin{cases} 2(1 + a_i^2)g_{ij} - 2a_i a_j g_{ji} = d_{1ij} + d_{2ij} + a_i(f_{ij} - f_{ji}), \\ 2(1 + a_j^2)g_{ji} - 2a_i a_j g_{ij} = d_{1ji} + d_{2ji} + a_j(f_{ji} - f_{ij}). \end{cases}$$

Thus

$$g_{ij} = \frac{1}{2(1 + a_i^2 + a_j^2)}((1 + a_j^2)(d_{1ij} + d_{2ij}) + a_i a_j(d_{1ji} + d_{2ji}) + a_i(f_{ij} - f_{ji})).$$

Let $\Phi = (\varphi_{ij}) \in R^{n\times n}$, $\varphi_{ij} = 1/2(1 + a_i^2 + a_j^2)$. Therefore, the solution of this minimization problem is

$$\hat{G} = \Phi * (D_1 + D_2 + D_1\Lambda^2 + D_2\Lambda^2 + \Lambda D_1^{\mathrm{T}}\Lambda + \Lambda D_2^{\mathrm{T}}\Lambda + \Lambda(F - F^{\mathrm{T}})). \qquad \square$$

We now claim that the notes we will use next are the same as in Theorem 2.1.

Theorem 3.1. *Given $X^* \in R^{n\times n}$, suppose matrices A, B, X_0 make Problem I solvable, let*

$$U^{\mathrm{T}} X^* U = \begin{pmatrix} X_{11}^* & X_{12}^* \\ X_{21}^* & X_{22}^* \end{pmatrix}, \tag{3.1}$$

$$P^{\mathrm{T}} X_{12}^* Q = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix}, \tag{3.2}$$

$$-P^{\mathrm{T}} X_{21}^{*\mathrm{T}} Q = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}, \tag{3.3}$$

$$Q^{\mathrm{T}} X_{22}^* Q = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}, \tag{3.4}$$

where $X_{11}^* \in R^{r\times r}$, $X_{22}^* \in R^{(n-r)\times(n-r)}$, $P^{\mathrm{T}} X_{12}^* Q$ and $P^{\mathrm{T}} X_{21}^{*\mathrm{T}} Q$ are partitioned according to $P^{\mathrm{T}} X_{12} Q$ in (2.12), $Q^{\mathrm{T}} X_{22}^* Q$ is partitioned according to $Q^{\mathrm{T}} X_{22} Q$. Then Problem II has an unique solution which can be expressed as

$$\hat{X} = U \begin{pmatrix} \Sigma^{-1} V_1^{\mathrm{T}} B V_1 \Sigma^{-1} & \hat{X}_{12} \\ -\hat{X}_{12}^{\mathrm{T}} & \hat{X}_{22} \end{pmatrix} U^{\mathrm{T}}, \tag{3.5}$$

$$\hat{X}_{12} = P \begin{pmatrix} \frac{1}{2}(E_{11}+F_{11}) & \tilde{X}_{12}\Omega_2^{-1} & \tilde{X}_{13} \\ \frac{1}{2}(E_{21}+F_{21}) & \hat{Y}_{22} & \hat{Y}_{23} \\ \frac{1}{2}(E_{31}+F_{31}) & \frac{1}{2}(E_{32}+F_{32}) & \frac{1}{2}(E_{33}+F_{33}) \end{pmatrix} Q^{\mathrm{T}},$$

$$\hat{X}_{22} = Q \begin{pmatrix} \frac{1}{2}(G_{11}-G_{11}^{\mathrm{T}}) & \frac{1}{2}(G_{12}-G_{21}^{\mathrm{T}}) & \frac{1}{2}(G_{13}-G_{31}^{\mathrm{T}}) \\ \frac{1}{2}(G_{21}-G_{12}^{\mathrm{T}}) & \Omega_2^{-1}(\tilde{X}_{22}-\Omega_1\hat{Y}_{22}\Omega_2+\Omega_2\hat{Y}_{22}^{\mathrm{T}}\Omega_1)\Omega_2^{-1} & \Omega_2^{-1}(\tilde{X}_{23}-\Omega_1\hat{Y}_{23}) \\ \frac{1}{2}(G_{31}-G_{13}^{\mathrm{T}}) & (\hat{Y}_{23}^{\mathrm{T}}\Omega_1-\tilde{X}_{23}^{\mathrm{T}})\Omega_2^{-1} & \tilde{X}_{33} \end{pmatrix} Q^{\mathrm{T}},$$

where $\hat{Y}_{22}=\Phi_1 * [E_{22}+F_{22}+(E_{22}+F_{22})\Omega_2^{-2}\Omega_1^2+\Omega_2^{-1}\Omega_1(E_{22}^{\mathrm{T}}+F_{22}^{\mathrm{T}})\Omega_2^{-1}\Omega_1+\Omega_2^{-2}\Omega_1(\tilde{X}_{22}-\tilde{X}_{22}^{\mathrm{T}})\Omega_2^{-1}+\Omega_2^{-1}\Omega_1(G_{22}^{\mathrm{T}}-G_{22})]$, $\hat{Y}_{23} = \Phi_2 * [E_{23} + F_{23} + 2\Omega_2^{-2}\Omega_1\tilde{X}_{23} + \Omega_2^{-1}\Omega_1(G_{32}^{\mathrm{T}} - G_{23})]$, $\Phi_1 = (\varphi_{1ij}) \in R^{l\times l}$, $\varphi_{1ij} = b_i^2 b_j^2/2(b_i^2 b_j^2 + a_i^2 b_j^2 + a_j^2 b_i^2)$, $\Phi_2 = (\varphi_{2ij}) \in R^{l\times(t-s-l)}$, $\varphi_{2ij} = b_i^2/2(a_i^2 + b_i^2)$.

Proof. Since matrices A, B, X_0 make Problem I solvable, the solution set S_E of Problem I is nonempty. And it can be easily proved to be a closed convex set. Hence Problem II has an unique solution $\hat{X} \in S_E$. From (2.7), (3.1) and the orthogonal invariance of Frobenius norm, it follows

$$
\begin{aligned}
\|X - X^*\|^2 &= \left\| U \begin{pmatrix} \Sigma^{-1} V_1^{\mathrm{T}} B V_1 \Sigma^{-1} & X_{12} \\ -X_{12}^{\mathrm{T}} & X_{22} \end{pmatrix} U^{\mathrm{T}} - X^* \right\|^2 \\
&= \|\Sigma^{-1} V_1^{\mathrm{T}} B V_1 \Sigma^{-1} - X_{11}^*\|^2 + \|X_{12} - X_{12}^*\|^2 + \|X_{12} + X_{21}^{*\mathrm{T}}\|^2 + \|X_{22} - X_{22}^*\|^2 \\
&= \left\| P \begin{pmatrix} Y_{11} & \tilde{X}_{12}\Omega_2^{-1} & \tilde{X}_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} Q^{\mathrm{T}} - X_{12}^* \right\|^2 + \left\| P \begin{pmatrix} Y_{11} & \tilde{X}_{12}\Omega_2^{-1} & \tilde{X}_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} Q^{\mathrm{T}} + X_{21}^{*\mathrm{T}} \right\|^2 \\
&\quad + \left\| Q \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ -Z_{12}^{\mathrm{T}} & \Omega_2^{-1}(\tilde{X}_{22} - \Omega_1 Y_{22} \Omega_2 + \Omega_2 Y_{22}^{\mathrm{T}} \Omega_1)\Omega_2^{-1} & \Omega_2^{-1}(\tilde{X}_{23} - \Omega_1 Y_{23}) \\ -Z_{13}^{\mathrm{T}} & (Y_{23}^{\mathrm{T}}\Omega_1 - \tilde{X}_{23}^{\mathrm{T}})\Omega_2^{-1} & \tilde{X}_{33} \end{pmatrix} Q^{\mathrm{T}} - X_{22}^* \right\|^2 \\
&\quad + \|\Sigma^{-1} V_1^{\mathrm{T}} B V_1 \Sigma^{-1} - X_{11}^*\|^2.
\end{aligned}
$$

By (3.2), (3.3), (3.4), $\|X - X^*\| = \min$ is equivalent to

$$
\begin{cases}
\|Y_{11} - E_{11}\|^2 + \|Y_{11} - F_{11}\|^2 = \min, \quad \|Y_{21} - E_{21}\|^2 + \|Y_{21} - F_{21}\|^2 = \min, \\
\|Y_{31} - E_{31}\|^2 + \|Y_{31} - F_{31}\|^2 = \min, \quad \|Y_{32} - E_{32}\|^2 + \|Y_{32} - F_{32}\|^2 = \min, \\
\|Y_{33} - E_{33}\|^2 + \|Y_{33} - F_{33}\|^2 = \min, \quad \|Z_{11} - G_{11}\|^2 = \min, \\
\|Z_{12} - G_{12}\|^2 + \|Z_{12} + G_{21}^{\mathrm{T}}\|^2 = \min, \quad \|Z_{13} - G_{13}\|^2 + \|Z_{13} + G_{31}^{\mathrm{T}}\|^2 = \min, \\
\|Y_{22} - E_{22}\|^2 + \|Y_{22} - F_{22}\|^2 + \|\Omega_2^{-1}(\tilde{X}_{22} - \Omega_1 Y_{22}\Omega_2 + \Omega_2 Y_{22}^{\mathrm{T}}\Omega_1)\Omega_2^{-1} - G_{22}\|^2 = \min, \\
\|Y_{23} - E_{23}\|^2 + \|Y_{23} - F_{23}\|^2 + \|\Omega_2^{-1}(\tilde{X}_{23} - \Omega_1 Y_{23}) - G_{23}\|^2 \\
\quad + \|(Y_{23}^{\mathrm{T}}\Omega_1 - \tilde{X}_{23}^{\mathrm{T}})\Omega_2^{-1} - G_{32}\|^2 = \min\,.
\end{cases}
$$

Notice that the last two equations above are equivalent to

$$
\begin{cases}
\|Y_{22} - E_{22}\|^2 + \|Y_{22} - F_{22}\|^2 + \|\Omega_2^{-1}\Omega_1 Y_{22} - Y_{22}^{\mathrm{T}}\Omega_1\Omega_2^{-1} - (\Omega_2^{-1}\tilde{X}_{22}\Omega_2^{-1} - G_{22})\|^2 = \min, \\
\|Y_{23} - E_{23}\|^2 + \|Y_{23} - F_{23}\|^2 + \|\Omega_2^{-1}\Omega_1 Y_{23} \\
\quad -(\Omega_2^{-1}\tilde{X}_{23} - G_{23})\|^2 + \|\Omega_2^{-1}\Omega_1 Y_{23} - (\Omega_2^{-1}\tilde{X}_{23} + G_{32}^{\mathrm{T}})\|^2 = \min\,.
\end{cases}
$$

So by Lemmas 3.1 and 3.2 we have

$$
\begin{cases}
Y_{11} = \frac{1}{2}(E_{11} + F_{11}), \quad Y_{21} = \frac{1}{2}(E_{21} + F_{21}), \quad Y_{31} = \frac{1}{2}(E_{31} + F_{31}), \quad Y_{32} = \frac{1}{2}(E_{32} + F_{32}), \\
Y_{33} = \frac{1}{2}(E_{33} + F_{33}), \quad Z_{11} = \frac{1}{2}(G_{11} - G_{11}^{\mathrm{T}}), \quad Z_{12} = \frac{1}{2}(G_{12} - G_{21}^{\mathrm{T}}),\ Z_{13} = \frac{1}{2}(G_{13} - G_{31}^{\mathrm{T}}), \\
Y_{22} = \Phi_1 * [E_{22} + F_{22} + (E_{22} + F_{22})\Omega_2^{-2}\Omega_1^2 + \Omega_2^{-1}\Omega_1(E_{22}^{\mathrm{T}} + F_{22}^{\mathrm{T}})\Omega_2^{-1}\Omega_1 \\
\qquad + \Omega_2^{-2}\Omega_1(\tilde{X}_{22} - \tilde{X}_{22}^{\mathrm{T}})\Omega_2^{-1} + \Omega_2^{-1}\Omega_1(G_{22}^{\mathrm{T}} - G_{22})], \\
Y_{23} = \Phi_2 * [E_{23} + F_{23} + 2\Omega_2^{-2}\Omega_1\tilde{X}_{23} + \Omega_2^{-1}\Omega_1(G_{32}^{\mathrm{T}} - G_{23})].
\end{cases}
$$

where $\Phi_1 = (\varphi_{1ij}) \in R^{l\times l}$, $\varphi_{1ij} = b_i^2 b_j^2 / 2(b_i^2 b_j^2 + a_i^2 b_j^2 + a_j^2 b_i^2)$, $\Phi_2 = (\varphi_{2ij}) \in R^{l\times(t-s-l)}$, $\varphi_{2ij} = b_i^2/2(a_i^2 + b_i^2)$.

Substituting $Y_{11}, Y_{21}, Y_{22}, Y_{23}, Y_{31}, Y_{32}, Y_{33}$ and Z_{11}, Z_{12}, Z_{13} into (2.7), we know the solution $\hat{X}$ of Problem II can be expressed as (3.5). $\square$

4. Numerical algorithm and an example

Based on the discussions above, a numerical algorithm for solving Problem II is given as follows:

(1) Input A, B, X_0, X^*;
(2) Make the SVD of A according to (2.1) and determine $U, V, U_i, V_i, i = 1, 2$;
(3) If (2.8) holds, continue, or go to step 11;
(4) Compute $\tilde{U}_1, \tilde{U}_2$ according to (2.2);
(5) Make the GSVD of matrix pair $[\tilde{U}_1^{\mathrm{T}}, \tilde{U}_2^{\mathrm{T}}]$ according to (2.3) and (2.4), and determine $P, Q, W, \Omega_1, \Omega_2$;
(6) Compute $\tilde{X}$ and partition it to determine $\tilde{X}_{ij} (i, j = 1, \ldots, 4)$ according to (2.5);
(7) If (2.14) holds, continue, or go to step 11;
(8) Partition $U^{\mathrm{T}} X^* U$ to determine X_{ij}^* $(i, j = 1, 2)$ according to (3.1);
(9) Partition $P^{\mathrm{T}} X_{12}^* Q, -P^{\mathrm{T}} X_{21}^{*\mathrm{T}} Q, Q^{\mathrm{T}} X_{22}^* Q$ according to (3.2), (3.3), (3.4), respectively, and determine $E_{ij}, F_{ij}, G_{ij}, i, j = 1, \ldots, 3$;
(10) Compute $\hat{X}$ according to (3.5);
(11) Stop.

Example. Let $n = 8, m = 4, q = 4$, given

$$A = \begin{pmatrix} 13.8912 & 4.5146 & -13.8912 & -3.4658 \\ 2.2857 & -9.1825 & -2.2857 & 18.9776 \\ -1.9735 & 6.6735 & 1.9735 & 1.3428 \\ 6.3856 & -18.6328 & -6.3856 & -6.8247 \\ 2.7287 & 8.4610 & -2.7287 & 30.2886 \\ -19.8816 & 5.6219 & 19.8816 & 5.4730 \\ 1.5340 & -20.2642 & -1.5340 & 15.0913 \\ 7.6842 & 6.2185 & -7.6842 & -6.7922 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -1.3703 & 0 & -6.8868 \\ 1.3703 & 0 & -1.3703 & 14.5589 \\ 0 & 1.3703 & 0 & 6.8868 \\ 6.8868 & -14.5589 & -6.8868 & 0 \end{pmatrix},$$

$$X_0 = \begin{pmatrix} 0 & 10.3215 & 5.1267 & 0.5107 \\ -10.3215 & 0 & 13.6660 & -6.6839 \\ -5.1267 & -13.6660 & 0 & -9.7160 \\ -0.5107 & 6.6839 & 9.7160 & 0 \end{pmatrix},$$

$$X^* = \begin{pmatrix} 8.3850 & -7.4813 & 8.5763 & 28.4428 & -4.2937 & 7.8322 & -15.4223 & 3.8720 \\ 5.8125 & -9.6836 & -7.3728 & 4.6994 & 22.5985 & 6.0867 & 23.5089 & 9.8377 \\ 3.7474 & 10.5292 & 13.6595 & 6.4863 & 5.7936 & -4.1125 & 8.7423 & -21.4078 \\ 7.2738 & -8.8137 & 11.1870 & 9.8332 & 7.6456 & 5.7886 & 12.1566 & 6.3557 \\ 5.6637 & 17.3050 & 8.9324 & 5.2850 & -15.9893 & 7.4236 & 7.6830 & 3.2589 \\ 4.4958 & -9.9773 & 19.9187 & 4.3554 & -6.0586 & 9.2698 & -9.5839 & 6.1765 \\ -9.4684 & 7.1448 & 22.8735 & 5.5569 & 20.9123 & -6.2976 & 19.1244 & 7.6631 \\ 2.1371 & 25.2364 & -6.1466 & 33.4820 & 3.9840 & 15.0312 & 7.8917 & -4.1247 \end{pmatrix}.$$

According to the above calculating steps and using MATLAB 5.3, we have $\hat{X}$ as follow:

$$\hat{X}=\begin{pmatrix} 0.0000 & 10.3215 & 5.1267 & 0.5107 & -6.2973 & 2.9509 & -6.3136 & 1.6110 \\ -10.3215 & -0.0000 & 13.6660 & -6.6839 & -1.1099 & 2.6268 & 10.1279 & -6.4440 \\ -5.1267 & -13.6660 & -0.0000 & -9.7160 & 0.4146 & -11.1976 & -5.5187 & -8.4876 \\ -0.5107 & 6.6839 & 9.7160 & 0.0000 & -4.3619 & -0.9443 & -1.8100 & -11.1097 \\ 6.2973 & 1.1099 & -0.4146 & 4.3619 & 0.0000 & 1.8384 & -0.4172 & -0.8296 \\ -2.9509 & -2.6268 & 11.1976 & 0.9443 & -1.8384 & 0.0000 & 4.7344 & -6.7377 \\ 6.3136 & -10.1279 & 5.5187 & 1.8100 & 0.4172 & -4.7344 & 0.0000 & 2.1986 \\ -1.6110 & 6.4440 & 8.4876 & 11.1097 & 0.8296 & 6.7377 & -2.1986 & 0.0000 \end{pmatrix}.$$

5. Conclusions

By using the decompositions of matrix(SVD and GSVD), the expansion problem of anti-symmetric matrix X under a linear constraint $A^{\mathrm{T}}XA=B$ is solved. Moreover, the optimal approximation solution is also obtained, an algorithm and a numerical example are provided.

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